

# NECESSARY AND SUFFICIENT CONDITIONS FOR LOCAL PARETO OPTIMALITY ON TIME SCALES

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**ABSTRACT.** We study a multiobjective variational problem on time scales. For this problem, necessary and sufficient conditions for weak local Pareto optimality are given. We also prove a necessary optimality condition for the isoperimetric problem with multiple constraints on time scales.

## 1. INTRODUCTION

The calculus on time scales was initiated by Aulbach and Hilger (see e.g. [2]) in order to create a theory that can unify discrete and continuous analysis. Since then, much active research has been observed all over the world (see e.g. [1, 3, 4, 7] and references therein). In this paper we consider multiobjective variational problems on time scales (Section 3.2). By developing a theory for multiobjective optimization problems on a time scale, one obtains more general results that can be applied to discrete, continuous or hybrid domains. To the best of the authors' knowledge, no study has been done in this field for time scales. The main results of the paper provide methods for identifying weak locally Pareto optimal solutions; versions for continuous domain one can find e.g. in [6, 8, 9]. We show that necessary optimality conditions for isoperimetric problems are also necessary for local Pareto optimality for a multiobjective variational problem on a time scale (Theorem 3.8), and the sufficient condition for local Pareto optimality can be reduced to the sufficient optimal condition for a basic problem of the calculus of variations on a time scale (Theorem 3.7). We also prove a necessary optimality condition for the isoperimetric problem with multiple constraints on time scales (Section 3.1).

## 2. TIME SCALES CALCULUS

In this section we introduce basic definitions and results that will be needed for the rest of the paper. For a more general theory of calculus on time scales, we refer the reader to [5].

A nonempty closed subset of  $\mathbb{R}$  is called a *time scale* and it is denoted by  $\mathbb{T}$ .

The *forward jump operator*  $\sigma : \mathbb{T} \rightarrow \mathbb{T}$  is defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the *backward jump operator*  $\rho : \mathbb{T} \rightarrow \mathbb{T}$  is defined by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T},$$

with  $\inf \emptyset = \sup \mathbb{T}$  (i.e.  $\sigma(M) = M$  if  $\mathbb{T}$  has a maximum  $M$ ) and  $\sup \emptyset = \inf \mathbb{T}$  (i.e.  $\rho(m) = m$  if  $\mathbb{T}$  has a minimum  $m$ ).

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A point  $t \in \mathbb{T}$  is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if  $\sigma(t) = t$ ,  $\sigma(t) > t$ ,  $\rho(t) = t$  and  $\rho(t) < t$ , respectively.

Throughout the paper we let  $\mathbb{T} = [a, b] \cap \mathbb{T}_0$  with  $a < b$  and  $\mathbb{T}_0$  a time scale containing  $a$  and  $b$ .

*Remark 2.1.* The time scales  $\mathbb{T}$  considered in this work have a maximum  $b$  and, by definition,  $\sigma(b) = b$ .

The *graininess function*  $\mu : \mathbb{T} \rightarrow [0, \infty)$  is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$

Following [5], we define  $\mathbb{T}^k = \mathbb{T} \setminus (\rho(b), b]$ ,  $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$ .

We say that a function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is *delta differentiable* at  $t \in \mathbb{T}^k$  if there exists a number  $f^\Delta(t)$  such that for all  $\varepsilon > 0$  there is a neighborhood  $U$  of  $t$  (i.e.  $U = (t - \delta, t + \delta) \cap \mathbb{T}$  for some  $\delta > 0$ ) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.$$

We call  $f^\Delta(t)$  the *delta derivative* of  $f$  at  $t$  and say that  $f$  is *delta differentiable* on  $\mathbb{T}^k$  provided  $f^\Delta(t)$  exists for all  $t \in \mathbb{T}^k$ .

For delta differentiable functions  $f$  and  $g$ , the next formula holds:

$$\begin{aligned} (fg)^\Delta(t) &= f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t) \\ &= f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t), \end{aligned}$$

where we abbreviate here and throughout the text  $f \circ \sigma$  by  $f^\sigma$ .

A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. We denote the set of all rd-continuous functions by  $C_{rd}$  and the set of all delta differentiable functions with rd-continuous derivative by  $C_{rd}^1$ .

It is known that rd-continuous functions possess an *antiderivative*, i.e. there exists a function  $F$  with  $F^\Delta = f$ , and in this case the *delta integral* is defined by  $\int_c^d f(t)\Delta t = F(c) - F(d)$  for all  $c, d \in \mathbb{T}$ . The delta integral has the following property:

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

We now present the integration by parts formulas for the delta integral:

**Lemma 2.2.** ([5]) *If  $c, d \in \mathbb{T}$  and  $f, g \in C_{rd}^1$ , then*

$$\begin{aligned} \int_c^d f(\sigma(t))g^\Delta(t)\Delta t &= [(fg)(t)]_{t=c}^{t=d} - \int_c^d f^\Delta(t)g(t)\Delta t; \\ \int_c^d f(t)g^\Delta(t)\Delta t &= [(fg)(t)]_{t=c}^{t=d} - \int_c^d f^\Delta(t)g(\sigma(t))\Delta t. \end{aligned}$$

We say that  $f : \mathbb{T} \rightarrow \mathbb{R}^n$  is a *rd-continuous* (a *delta differentiable*) function if each component of  $f$ ,  $f_i : \mathbb{T} \rightarrow \mathbb{R}$ , is a rd-continuous (a delta differentiable) function. By abuse of notation, we continue to write  $C_{rd}$  for the set of all rd-continuous vector valued functions and  $C_{rd}^1$  for the set of all delta differentiable vector valued functions with rd-continuous derivative.

The following Dubois-Reymond lemma for the calculus of variations on time scales will be useful for our purposes.

**Lemma 2.3.** (Lemma of Dubois-Reymond [4]) *Let  $g \in C_{rd}$ ,  $g : [a, b]^k \rightarrow \mathbb{R}^n$ . Then,*

$$\int_a^b g(t) \cdot \eta^\Delta(t)\Delta t = 0 \quad \text{for all } \eta \in C_{rd}^1 \text{ with } \eta(a) = \eta(b) = 0$$

if and only if  $g(t) = c$  on  $[a, b]^k$  for some  $c \in \mathbb{R}^n$ .

### 3. MAIN RESULTS

We begin by proving necessary optimality conditions for isoperimetric problems on time scales (§3.1). In §3.2 we show that Pareto solutions of multiobjective variational problems on time scales are minimizers of a certain family of isoperimetric problems on time scales.

#### 3.1. Isoperimetric problem on time scales.

**Definition 3.1.** For  $f : [a, b] \rightarrow \mathbb{R}^n$  we define the norm

$$\|f\|_{C_{rd}^1} = \max_{t \in [a, b]^k} \|f^\sigma(t)\| + \max_{t \in [a, b]^k} \|f^\Delta(t)\|,$$

where  $\|\cdot\|$  stands for any norm in  $\mathbb{R}^n$ .

Let  $\mathcal{L} : C_{rd}^1 \rightarrow \mathbb{R}$  be a functional defined on the function space  $C_{rd}^1$  endowed with the norm  $\|\cdot\|_{C_{rd}^1}$  and let  $A \subseteq C_{rd}^1$ .

**Definition 3.2.** A function  $\hat{f} \in A$  is called a *weak local minimum* of  $\mathcal{L}$  provided there exists  $\delta > 0$  such that  $\mathcal{L}[\hat{f}] \leq \mathcal{L}[f]$  for all  $f \in A$  with  $\|f - \hat{f}\|_{C_{rd}^1} < \delta$ .

Now, let us consider a functional of the form

$$(1) \quad \mathcal{L}[y] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t,$$

where  $a, b \in \mathbb{T}$  with  $a < b$ ,  $L(t, s, v) : [a, b]^k \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  has partial continuous derivatives with respect to the second and third variables for all  $t \in [a, b]^k$ , and  $L(t, \cdot, \cdot)$  and its partial derivatives are rd-continuous at  $t$ . The *isoperimetric problem* consists of finding a function  $y$  satisfying:

(i) the boundary conditions

$$(2) \quad y(a) = \alpha, \quad y(b) = \beta, \quad \alpha, \beta \in \mathbb{R}^n;$$

and

(ii) constraints of the form

$$(3) \quad \mathcal{G}_i[y] = \int_a^b G_i(t, y^\sigma(t), y^\Delta(t)) \Delta t = \xi_i, \quad i = 1, \dots, m,$$

where  $\xi_i, i = 1, \dots, m$ , are specified real constrains,  $G_i(t, s, v) : [a, b]^k \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, m$ , have partial continuous derivatives with respect to the second and third variables for all  $t \in [a, b]^k$ , and  $G_i(t, \cdot, \cdot)$  and their partial derivatives are rd-continuous at  $t$ ; that takes (1) to a minimum.

**Definition 3.3.** Let  $\mathcal{L}$  be a functional defined on  $C_{rd}^1$ . The first variation of  $\mathcal{L}$  at  $y \in C_{rd}^1$  in the direction  $\eta \in C_{rd}^1$ , also called *Gâteaux derivative* with respect to  $\eta$  at  $y$ , is defined as

$$\delta \mathcal{L}[y; \eta] = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{L}[y + \varepsilon \eta] - \mathcal{L}[y]}{\varepsilon} = \frac{\partial}{\partial \varepsilon} \mathcal{L}[y + \varepsilon \eta]|_{\varepsilon=0}$$

(provided it exists). If the limit exists for all  $\eta \in C_{rd}^1$ , then  $\mathcal{L}$  is said to be *Gâteaux differentiable* at  $y$ .

The existence of Gâteaux derivative  $\delta \mathcal{L}[y; \eta]$  presupposes that:

- (i)  $\mathcal{L}[y]$  is defined;
- (ii)  $\mathcal{L}[y + \varepsilon \eta]$  is defined for all sufficiently small  $\varepsilon$ .

**Theorem 3.4.** *Let  $\mathcal{L}, \mathcal{G}_1, \dots, \mathcal{G}_m$  be functionals defined in a neighborhood of  $\hat{y}$  and having continuous Gâteaux derivative in this neighborhood. Suppose that  $\hat{y}$  is a weak local minimum of (1) subject to the boundary conditions (2) and the isoperimetric constrains (3). Then, either:*

(i)  $\forall v_j \in C_{rd}^1, j = 1, \dots, m$

$$(4) \quad \begin{vmatrix} \delta \mathcal{G}_1[\hat{y}; v_1] & \delta \mathcal{G}_1[\hat{y}; v_2] & \cdots & \delta \mathcal{G}_1[\hat{y}; v_m] \\ \delta \mathcal{G}_2[\hat{y}; v_1] & \delta \mathcal{G}_2[\hat{y}; v_2] & \cdots & \delta \mathcal{G}_2[\hat{y}; v_m] \\ \vdots & \vdots & \ddots & \vdots \\ \delta \mathcal{G}_m[\hat{y}; v_1] & \delta \mathcal{G}_m[\hat{y}; v_2] & \cdots & \delta \mathcal{G}_m[\hat{y}; v_m] \end{vmatrix} = 0$$

or

(ii) there exist constants  $\lambda_i \in \mathbb{R}, i = 1, \dots, m$  for which

$$(5) \quad \delta \mathcal{L}[\hat{y}; \eta] = \sum_{i=1}^m \lambda_i \delta \mathcal{G}_i[\hat{y}; \eta] \quad \forall \eta \in C_{rd}^1.$$

*Proof.* This proof is patterned after the proof of Troutman [10, Theorem 5.16]. Let us consider, for fixed directions  $\eta, v_1, v_2, \dots, v_m$ , the auxiliary functions:

$$\begin{aligned} l(p, q_1, \dots, q_m) &= \mathcal{L}[\hat{y} + p\eta + q_1 v_1 + \cdots + q_m v_m], \\ g_1(p, q_1, \dots, q_m) &= \mathcal{G}_1[\hat{y} + p\eta + q_1 v_1 + \cdots + q_m v_m], \\ &\vdots \\ g_m(p, q_1, \dots, q_m) &= \mathcal{G}_m[\hat{y} + p\eta + q_1 v_1 + \cdots + q_m v_m], \end{aligned}$$

which are defined in some neighborhood of the origin in  $\mathbb{R}^{m+1}$ , since  $\mathcal{L}, \mathcal{G}_1, \dots, \mathcal{G}_m$  themselves are defined in a neighborhood of  $\hat{y}$ . Note that the partial derivative

$$\begin{aligned} l_p(p, q_1, \dots, q_m) &= \frac{\partial}{\partial p} l(p, q_1, \dots, q_m) = \frac{\partial}{\partial p} \mathcal{L}[\hat{y} + p\eta + q_1 v_1 + \cdots + q_m v_m] \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{L}[\hat{y} + (p + \varepsilon)\eta + q_1 v_1 + \cdots + q_m v_m] - \mathcal{L}[\hat{y} + p\eta + q_1 v_1 + \cdots + q_m v_m]}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{L}[y + \varepsilon \eta] - \mathcal{L}[y]}{\varepsilon}, \end{aligned}$$

with  $y = \hat{y} + p\eta + q_1 v_1 + \cdots + q_m v_m$ . Therefore,  $l_p(p, q_1, \dots, q_m) = \delta \mathcal{L}[\hat{y}; \eta]$ . Similarly we have:

$$\begin{aligned} l_{q_i}(p, q_1, \dots, q_m) &= \delta \mathcal{L}[\hat{y}; v_i], \quad i = 1, \dots, m, \\ (g_j)_p(p, q_1, \dots, q_m) &= \delta \mathcal{G}_j[\hat{y}; \eta], \quad j = 1, \dots, m, \\ (g_j)_{q_i}(p, q_1, \dots, q_m) &= \delta \mathcal{G}_j[\hat{y}; v_i], \quad i = 1, \dots, m, \quad j = 1, \dots, m. \end{aligned}$$

Hence, the Jacobian determinant  $\frac{\partial(l, g_1, \dots, g_m)}{\partial(p, q_1, \dots, q_m)}$  evaluated at  $(p, q_1, \dots, q_m) = (0, 0, \dots, 0)$  is the following:

$$(6) \quad \begin{vmatrix} \delta \mathcal{L}[\hat{y}; \eta] & \delta \mathcal{L}_1[\hat{y}; v_1] & \cdots & \delta \mathcal{L}_1[\hat{y}; v_m] \\ \delta \mathcal{G}_1[\hat{y}; \eta] & \delta \mathcal{G}_1[\hat{y}; v_1] & \cdots & \delta \mathcal{G}_1[\hat{y}; v_m] \\ \vdots & \vdots & \ddots & \vdots \\ \delta \mathcal{G}_m[\hat{y}; \eta] & \delta \mathcal{G}_m[\hat{y}; v_1] & \cdots & \delta \mathcal{G}_m[\hat{y}; v_m] \end{vmatrix}.$$

Note also that the vector valued function  $(l, g_1, \dots, g_m)$  has continuous partial derivatives in a neighborhood of the origin, since  $\mathcal{L}, \mathcal{G}_1, \dots, \mathcal{G}_m$  have continuous Gâteaux derivative in the neighborhood of  $\hat{y}$ . With this preparation we can prove our theorem. Assume condition (i) does not hold for one set of directions:  $v_1, v_2, \dots, v_m$  and suppose there exists one direction  $\eta$  for which the determinant (6) is nonvanishing. Therefore, the classical inverse function theorem applies, i.e. the application  $(l, g_1, \dots, g_m)$  maps a neighborhood of the origin in  $\mathbb{R}^{m+1}$  onto a region

containing a full neighborhood of  $(\mathcal{L}[\hat{y}], \mathcal{G}_1[\hat{y}], \dots, \mathcal{G}_m[\hat{y}])$ . That is, one can find pre-image points  $(\hat{p}, \hat{q}_1, \dots, \hat{q}_m)$  and  $(\hat{p}, \hat{q}_1, \dots, \hat{q}_m)$  near the origin, for which the points  $\hat{y} = \hat{y} + \hat{p}\eta + \sum_{i=1}^m \hat{q}_i v_i$  and  $\hat{y} = \hat{y} + \hat{p}\eta + \sum_{i=1}^m \hat{q}_i v_i$  satisfy the conditions:

$$\begin{aligned} \mathcal{L}[\hat{y}] &< \mathcal{L}[\hat{y}] < \mathcal{L}[\hat{y}], \\ \mathcal{G}_i[\hat{y}] &= \mathcal{G}_i[\hat{y}] = \mathcal{G}_i[\hat{y}], \quad i = 1, \dots, m. \end{aligned}$$

This shows that  $\hat{y}$  cannot be a local extremal for  $\mathcal{L}$  subject to constraints (3), contradicting the hypothesis. Thus, for the specific set of directions:  $v_1, v_2, \dots, v_m$  the determinant (6) must vanish for each  $\eta \in C_{rd}^1$ . We expand it by minors of the first column

$$(7) \quad \delta\mathcal{L}[\hat{y}; \eta] \cdot \text{cof} \delta\mathcal{L}[\hat{y}; \eta] + \delta\mathcal{G}_1[\hat{y}; \eta] \cdot \text{cof} \delta\mathcal{G}_1[\hat{y}; \eta] + \dots + \delta\mathcal{G}_m[\hat{y}; \eta] \cdot \text{cof} \delta\mathcal{G}_m[\hat{y}; \eta] = 0,$$

where we are using the notation *cof* to denote the cofactor. Dividing equation (7) by  $\text{cof} \delta\mathcal{L}[\hat{y}; \eta]$ , since it is precisely the nonvanishing determinant  $\left| \begin{array}{c} \delta\mathcal{G}_i[\hat{y}; v_j] \\ i, j = 1, \dots, m \end{array} \right|$ , and setting

$$\lambda_i = -\frac{\text{cof} \delta\mathcal{G}_i[\hat{y}; \eta]}{\text{cof} \delta\mathcal{L}[\hat{y}; \eta]}$$

we obtain an equation equivalent to (5).  $\square$

Note that condition (ii) of Theorem 3.4 can be written in the form

$$(8) \quad \delta \left( \mathcal{L} - \sum_{i=1}^m \lambda_i \mathcal{G}_i[\hat{y}; \eta] \right) = 0 \quad \forall \eta \in C_{rd}^1,$$

since the Gâteaux derivative is a linear operation on the functionals (by the linearity of the ordinary derivative).

Now, suppose that assumptions of Theorem 3.4 hold but condition (i) does not hold. Then, equation (8) is fulfilled for every  $\eta \in C_{rd}^1$ . Let us consider function  $\eta$  such that  $\eta(a) = \eta(b) = 0$  and denote by  $\mathcal{F}$  the functional  $\mathcal{L} - \sum_{i=1}^m \lambda_i \mathcal{G}_i$ . Then we have

$$\begin{aligned} 0 &= \delta\mathcal{F}[\hat{y}; \eta] = \frac{\partial}{\partial \varepsilon} \mathcal{F}[\hat{y} + \varepsilon\eta]|_{\varepsilon=0} \\ &= \int_a^b (F_s(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) \eta^\sigma(t) + F_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) \eta^\Delta(t)) \Delta t, \end{aligned}$$

where the function  $F : [a, b]^k \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is defined by  $F(t, s, v) = L(t, s, v) - \sum_{i=1}^m \lambda_i G_i(t, s, v)$ . Note that

$$\int_a^b \left( \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \eta(t) \right)^\Delta \Delta t = \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \eta(t) \Big|_{t=a}^{t=b} = 0$$

and

$$\begin{aligned} &\int_a^b \left( \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \eta(t) \right)^\Delta \Delta t \\ &= \int_a^b \left\{ \left( \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \right)^\Delta \eta^\sigma(t) + \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \eta^\Delta(t) \right\} \Delta t \\ &= \int_a^b \left\{ F_s(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) \eta^\sigma(t) + \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \eta^\Delta(t) \right\} \Delta t. \end{aligned}$$

Therefore,

$$0 = \int_a^b \left\{ F_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) - \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta \tau \right\} \eta^\Delta(t) \Delta t.$$

Since the function  $\eta$  is arbitrary, Lemma 2.3 implies that

$$F_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) - \int_a^t F_s(\tau, \hat{y}^\sigma(\tau), \hat{y}^\Delta(\tau)) \Delta\tau = c$$

for some  $c \in \mathbb{R}^n$  and all  $t \in [a, b]^k$ . Hence,

$$(9) \quad F_v^\Delta(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) = F_s(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$$

for all  $t \in [a, b]^{k^2}$ .

We have just proved the following necessary optimality condition for the isoperimetric problem with multiple constraints on time scales.

**Theorem 3.5.** *Let us assume the assumptions of Theorem 3.4 hold but condition (4) does not hold. If  $\hat{y} \in C_{rd}^1$  is a weak local minimum of the problem (1)-(3), then it satisfies the Euler-Lagrange equation (9) for all  $t \in [a, b]^{k^2}$ .*

**3.2. Pareto optimality.** Let us consider a finite number  $d \geq 1$  of (objective) functionals:

$$(10) \quad \mathcal{L}_i[y] = \int_a^b L_i(t, y^\sigma(t), y^\Delta(t)) \Delta t, \quad i = 1, \dots, d,$$

where  $a, b \in \mathbb{T}$  with  $a < b$ ,  $L_i(t, s, v) : [a, b]^k \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i = 1, \dots, d$ , have partial continuous derivatives with respect to the second and third variables for all  $t \in [a, b]^k$ , and  $L_i(t, \cdot, \cdot)$  and their partial derivatives,  $i = 1, \dots, d$ , are rd-continuous at  $t$ . We would like to find a function  $y \in C_{rd}^1$ , satisfying the boundary conditions (2), that renders the minimum value to each functional  $\mathcal{L}_i$ ,  $i = 1, \dots, d$ , simultaneously. In general, there does not exist such a function, and one uses the concept of Pareto optimality.

**Definition 3.6.** A function  $\hat{y} \in C_{rd}^1$  is called a *weak locally Pareto optimal solution* if there exists  $\delta > 0$  such that there does not exist  $y \in C_{rd}^1$  with  $\|y - \hat{y}\|_{C_{rd}^1} < \delta$  and

$$\forall i \in \{1, \dots, d\} : \mathcal{L}_i[y] \leq \mathcal{L}_i[\hat{y}] \wedge \exists j \in \{1, \dots, d\} : \mathcal{L}_j[y] < \mathcal{L}_j[\hat{y}].$$

**Theorem 3.7.** *If  $\hat{y}$  is a weak local minimum of the functional  $\sum_{i=1}^d \gamma_i \mathcal{L}_i[y]$  with  $\gamma_i > 0$  for  $i = 1, \dots, d$  and  $\sum_{i=1}^d \gamma_i = 1$ , then it is a weak locally Pareto optimal solution of the multiobjective problem with functionals (10).*

*Proof.* Let  $\hat{y}$  be a weak local minimum of the functional  $\sum_{i=1}^d \gamma_i \mathcal{L}_i[y]$  with  $\gamma_i > 0$  for  $i = 1, \dots, d$  and  $\sum_{i=1}^d \gamma_i = 1$ . Suppose on the contrary that  $\hat{y}$  is not a weak locally Pareto optimal. Then, for every  $\delta > 0$  there exists  $y$  with  $\|y - \hat{y}\|_{C_{rd}^1} < \delta$  such that  $\forall i \in \{1, \dots, d\}$  we have  $\mathcal{L}_i[y] \leq \mathcal{L}_i[\hat{y}]$  and  $\exists j \in \{1, \dots, d\}$  such that  $\mathcal{L}_j[y] < \mathcal{L}_j[\hat{y}]$ . Since  $\gamma_i > 0$  for  $i = 1, \dots, d$ , we obtain  $\sum_{i=1}^d \gamma_i \mathcal{L}_i[y] < \sum_{i=1}^d \gamma_i \mathcal{L}_i[\hat{y}]$ . This contradicts our choice of  $\hat{y}$ .  $\square$

**Theorem 3.8.** *If  $\hat{y}$  is a weak locally Pareto optimal solution of the multiobjective problem with functionals (10), then it minimizes each one of the scalar functionals*

$$\mathcal{L}_i[y], \quad i \in \{1, \dots, d\}$$

*subject to the constraints*

$$\mathcal{L}_j[y] = \mathcal{L}_j[\hat{y}], \quad j = 1, \dots, d \text{ and } j \neq i.$$

*Proof.* Let  $\hat{y}$  be a weak locally Pareto optimal solution of the problem on time scales (10) and suppose the contrary, i.e. that for some  $i$   $\hat{y}$  does not solve the problem  $\mathcal{L}_i[y] \rightarrow \min$  subject to  $\mathcal{L}_j[y] = \mathcal{L}_j[\hat{y}], j = 1, \dots, d (j \neq i)$ . Then, for every  $\delta > 0$  there exists  $y$  with  $\|y - \hat{y}\|_{C_{rd}^1} < \delta$  such that  $\mathcal{L}_i[y] < \mathcal{L}_i[\hat{y}]$  and  $\mathcal{L}_j[y] = \mathcal{L}_j[\hat{y}], j = 1, \dots, d (j \neq i)$ . This contradicts the weak local Pareto optimality of  $\hat{y}$ .  $\square$

**Example 3.9.** Let  $\mathbb{T} = \{0, 1, 2\}$ . We would like to find locally Pareto optimal solutions for

$$\begin{aligned}\mathcal{L}_1[y] &= \int_0^2 y^2(t+1) \Delta t, \\ \mathcal{L}_2[y] &= \int_0^2 (y(t+1) - 2)^2 \Delta t\end{aligned}$$

satisfying the boundary conditions  $y(0) = 0$ ,  $y(2) = 0$ . Note that

$$\mathcal{L}_1[y] = \sum_{t=0}^1 y^2(t+1), \quad \mathcal{L}_2[y] = \sum_{t=0}^1 (y(t+1) - 2)^2,$$

and that the possible solutions are of the form

$$y(t) = \begin{cases} 0 & \text{if } t = 0 \\ a & \text{if } t = 1 \\ 0 & \text{if } t = 2, \end{cases}$$

where  $a \in \mathbb{R}$ . On account of the above, we have  $\mathcal{L}_1[y(t)] = a^2$ , and  $\mathcal{L}_2[y(t)] = 4 + (a - 2)^2$ . Using Theorem 3.7 we obtain that locally Pareto optimal solutions for functionals  $\mathcal{L}_1$ ,  $\mathcal{L}_2$  are

$$y(t) = \begin{cases} 0 & \text{if } t = 0 \\ a & \text{if } t = 1, \\ 0 & \text{if } t = 2 \end{cases}, \quad a \in [0, 2].$$

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